I. Random Concepts Review

- Concept of probability, event, random variable (RV)
- Concept & properties of cumulative distribution function (CDF), probability density function (PDF)
- Mean, moment, skewness, kurtosis
- Useful RV: uniform, Gaussian
- Random vector definition, characterization, statistical description
- Correlation/Covariance matrices: definition & properties
- Cross-correlation, cross-covariance matrices: definition & properties
- Random vector linear transformation
- Central limit theorem

I. Random Concepts Review

❖ <u>What is a Probability</u>:

❖ <u>What is an Event</u>:

❖ <u>What is a Random Variable (RV)</u>:

<u>Example</u>: Coin tossed 3 times

* sample space $S = \{ \qquad \}$

❖ Cumulative Distribution Function (cdf):

❖ CDF Properties:

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-
-
-

❖ Probability Density Function: $f_x(x) =$

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-

- RV completely characterized by pdf
- pdf information can be summarized by key aspects called statistical averages or moments

(1) <u>mean/average</u>

- $E\{x\} = m_x =$ if x is discrete

  $=$ if x is continuous

- important property of the mean $\rightarrow$ linearity!

  $\rightarrow E\{\alpha x + \beta\} =$

  $\rightarrow E\{g(x)\} =$

(3) <u>moments</u>

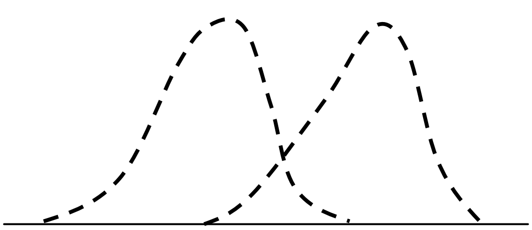
- $r_x^{(m)} = E\left\{x^m\right\} =$
- $\sigma_x^{(m)} = E\left\{\left|x - m_x\right|^m\right\} =$
- variance = $\sigma_x^{(2)}$
- variance property: $\sigma_x^2 = E\left[\left|x\right|^2\right] - \left(E\left[x\right]\right)^2$
  - proof:

❖ Useful Moments:

Skewness

measures degree of asymmetry of distribution around the mean

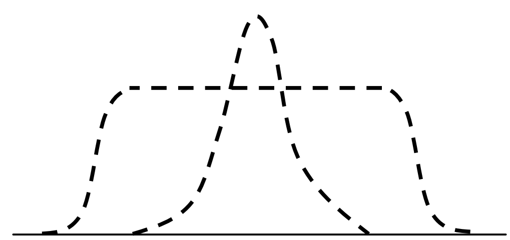
$$k_x^{(3)} = E\left\{\left(\frac{x-m_x}{\sigma_x}\right)^3\right\} =$$



Kurtosis

measures relative flatness or peakedness of distribution about its mean

$$k_x^{(4)} = E\left\{\left(\frac{x-m_x}{\sigma_x}\right)^4\right\} - 3$$



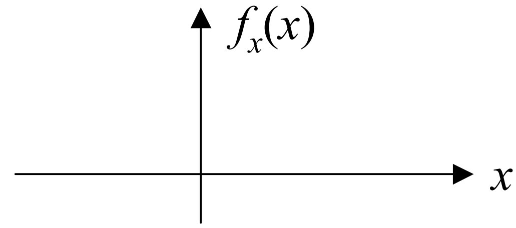
Note:

→ $k_x^{(4)} = 0$ for normal distribution

❖ <u>Useful RVs</u>:

(1) Uniform RV

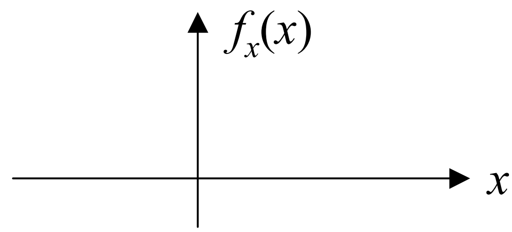
$f_x(x) =$



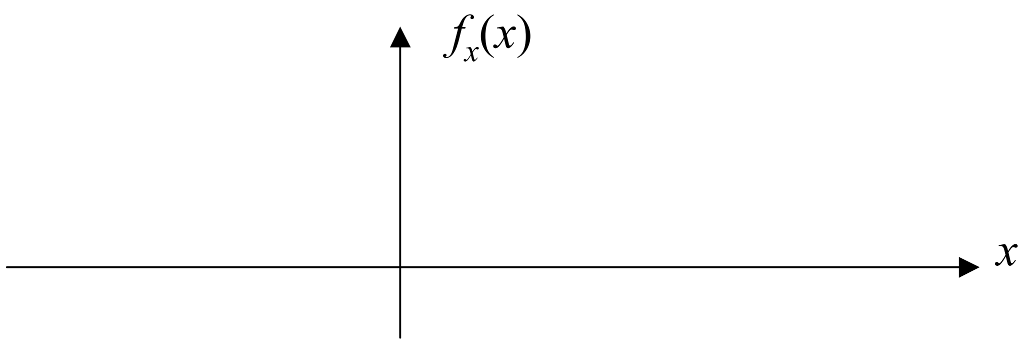
<u>mean/variance</u>:

(1) Normal RV

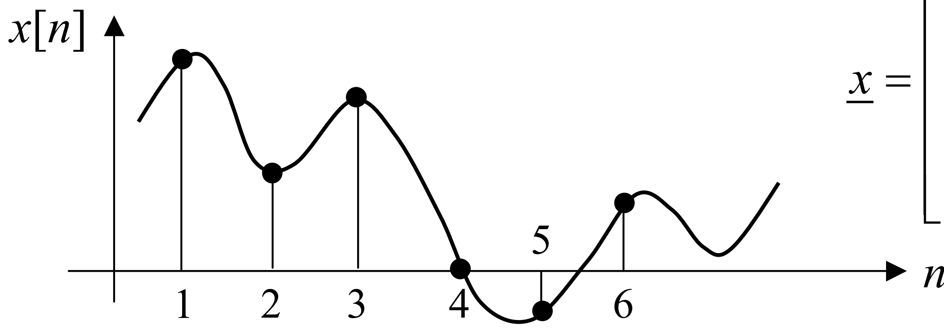
$f_x(x) =$



<u>Property</u>



❖ <u>Random Vector</u>:



$$\underline{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

- Group of signal observations can be modeled as a collection of RVs that can be grouped together to form a random vector.

- <u>Random Vector Distribution</u>:

  → Random vector completely defined by its joint distribution function.

- <u>Random Vector Density</u>:

❖ Complex Random Vector:

- Complex Random Variable:

- Complex Random Variable Mean /Variance:

- Complex Random Vector:

❖ Random Vector Statistical Description:

(1) Mean Vector:

$$\underline{m}_{\underline{x}} = E\left\{\underline{x}\right\} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

where:

(2) <u>Correlation/Covariance Matrices</u>:

- Correlation matrix

$$R_{\underline{x}} = E\left\{\underline{x}\cdot\underline{x}^H\right\} = E\left\{ \qquad \right\}$$

$$=$$

$$E\left\{\left|x_i\right|^2\right\} =$$

$$E\left\{x_i\cdot x_j^{\,*}\right\} =$$

- Covariance matrix

$$C_{\underline{n}} = E\left\{\left(\underline{x}-\underline{m}_{\underline{x}}\right)\left(\underline{x}-\underline{m}_{\underline{x}}\right)^H\right\}$$

$$=$$

❖ Covariance/Correlation Matrices are related:

$$C_{\underline{x}} = R_{\underline{x}} - m_{\underline{x}} m_{\underline{x}}^H$$

- *Proof:*

❖ Correlation Matrix Properties:

(1) Conjugate symmetry

(2) Positive semi-definite

Proofs:

- Eigendecomposition and PSD (positive semi-definite)

  Matrix A is said to be PSD iff $\lambda(A) \geq 0$

  Matrix A is said to be PD (positive definite)

  iff $(A) > 0$

- Example: $B = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ PSD?

❖ Cross-Correlation and Covariance Matrix:

- $R_{xy} = E\left[\underline{x}\underline{y}^H\right]$
- $C_{xy} = E\left\{\left(\underline{x} - \underline{m}_{\underline{x}}\right)\left(\underline{y} - \underline{m}_{\underline{y}}\right)^H\right\}$
- $C_{xy} = R_{xy} - \underline{m}_{\underline{x}} \cdot \underline{m}_{\underline{y}}^H$
- Properties:

(1) 2 random vectors are said to be *uncorrelated* if:

(2) 2 random vectors are said to *orthogonal* if:

(3) when $\underline{m}_{\underline{x}} = \underline{0}$ or $\underline{m}_{\underline{y}} = \underline{0}$

$\underline{x}$ & $\underline{y}$ *uncorrelated* $\Rightarrow$ $\underline{x}$ & $\underline{y}$ *orthogonal*

(4) if a vector has orthogonal components, then

$$E\left\{x_i x_j^*\right\} = 0 \quad \text{when} \quad i \neq j$$

$$\Rightarrow \quad R_{\underline{x}} =$$

•Note: "correlatedness" is different from independence

•Recall: to check independence:
if $x_1(\xi)$ and $x_2(\xi)$ are independent

$$f_{x_1,x_2}(x_1, x_2) = f_{x_1}(x_1).f_{x_2}(x_2)$$

$$=> E\{x_1(\xi)x_2(\xi)\} = E\{x_1(\xi)\}E\{x_2(\xi)\}$$

<u>Consequence</u>: independence uncorrelated

❖ Normal Random Vector:

(1) Real random vector

(2) Complex random vector

(3) Important properties of normal random vectors

- pdf completely specified by mean + matrices
- if components of $\underline{x}$ are mutually uncorrelated $\Rightarrow$ they are independent
- if $\underline{x}$ is normal $\Rightarrow \underline{y} = A\underline{x}$ is normal

❖ Linear Transformation for Random Vectors:

$\rightarrow \quad \underline{y} = A\underline{x}$

- Mean vector

$$E\left[\underline{y}\right] =$$

- Correlation matrix

$$R_y = E\left[\underline{y}\,\underline{y}^H\right] =$$

- Covariance matrix

$$C_y =$$

❖Central Limit Theorem (CLT):

Describes the limiting behavior of the distribution function of a normalized sum of I.I.D. variables

Define:

$$z_n = \frac{s_n - nm}{\sigma\sqrt{n}}$$

$$where \quad s_n = \sum_{i=1}^{n} x_i;\ m = E[x_i];\ \sigma^2 = \text{var}[x_i]$$

As n gets large, $z_n \sim N(0,1)$

As n gets large, $s_n \sim N(nm, n\sigma^2)$

Example: Application of the CLT

Suppose orders at a restaurant are IID with a mean price m=\$8.00 and standard deviation σ=\$2.00.
Estimate the probability that the first 100 customers spend a total of more than \$840.00